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METHOD OF CANONICAL ELEMENTS FOR MODELING TRANSFER PROCESSES IN MULTIPLY CONNECTED REGIONS OF AN ARBITRARY SHAPE

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A refined method of canonical elements for calculation of processes of heat and mass transfer and deformation in multiply connected bodies of a complex shape with curvilinear boundaries is stated. Results of comparison of the data of numerical experiments with accurate analytical solutions are presented.

Development of numerical simulation and even wider use of it in solving urgent scientific-engineering problems requires the creation of universal, efficient, and, at the same time, rather simple calculation methods and algorithms with specified accuracy for transfer processes in multiply connected regions of an arbitrary shape with variable thermophysical characteristics and arbitrary boundary and initial conditions.

In [1, 2], a new approach to the solution of partial differential equations for regions of an arbitrary shape is suggested. The approach is based on approximation of the initial differential equation by a balance equation for an element of a canonical shape that is constructed on a nonuniform difference grid. In this case, difference derivatives of the sought scalar function along the coordinate axes are determined as projections of its gradient, which in turn is expressed in terms of derivatives along some axes passing through the nodes of the nonuniform grid. This approach, called the method of canonical elements, has certain advantages with respect to simplicity of algorithms and accuracy of the solution as compared to known numerical methods usually used for these problems, in particular, the method of finite elements, which is based on the search for an extremum of the functional corresponding to the initial differential equation.

In what follows we consider the problems of automation of construction of nonuniform difference grids for multiply connected regions of an arbitrary shape, a difference grid of elevated accuracy for the method of canonical elements, and the results of solution of some problems of transfer of energy and momentum in multiply connected deformable bodies with curvilinear boundaries.

The method of canonical elements can be implemented, generally speaking, on arbitrary nonuniform grids. However, to simplify the algorithms and to make them more universal and also to provide the possibility of automated construction of nonuniform difference grids, it is expedient to use regularized grids. Regularization of grids can be performed, in particular, by positioning nodal points on the walls of coordinate surfaces and straight lines. For a simply connected body in Cartesian coordinates, this grid is described by the equations

$$z_j = z_{j-1} + h_{zj-1}\,, \quad j = 0, 1, \ldots, J\,, \quad z_0 = z'\,, \quad z_J = z''\,;$$

$$y_{mj} = y_{m-1,j} + h_{ym-1,j}\,, \quad m = 0, 1, \ldots, M(j)\,, \quad y_{0j} = y'_j\,, \quad y_{Mj} = y''_j\,;$$

$$x_{imj} = x_{i-1,mj} + ih_{x,i-1,mj}\,, \quad i = 0, 1, \ldots, I(m, j)\,, \quad x_{0mj} = x'_{mj}\,, \quad x_{Imj} = x''_{mj}\,;$$

$$t_n = t_{n-1} + h_{tn-1}\,, \quad n = 1, 2, \ldots, \quad h_{tn} > 0\,, \quad t_0 = 0\,.$$

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Here z', y'_j, x'_{mj} are the minimum values of the coordinates, respectively: z for points of the region, y for points of the cross section z_j, and x for points of the intersection line of the coordinate surfaces $z = z_j$ and $y = y_{mj}$; z'', y''_j, x''_{mj} are the maximum values of the coordinates for the same elements of the region.

The quasi-uniform grid [1], which may be used in solving many practical problems, is the simplest case of a regularized grid. For this grid $h_{zj} = \text{const}$, $h_{ymj} = h_{yj} \neq f(m)$, $h_{imj} = h_{mj} \neq f(i)$, and $h_{tn} = \text{const}$.

An analysis of different versions of nonuniform difference grids in multiply connected systems of a complex configuration showed that from the point of view of simplicity of the algorithm of solution, its universality, and automation of construction of grids it is expedient to use quasi-uniform grids. By virtue of this a method is suggested which presuposes conventional division of a multiply connected region into a set of simply connected subregions. Boundary nodal points of each of them pertain only to this subregion. They lie on the outer and inner boundaries of the body or are at a distance of a mesh width along the coordinate axis from the node pertaining to a neighboring subregion.

In numerical simulation of transfer processes in regions of complex configuration the problem of approximation of partial derivatives on nonuniform grids is knotty. It is shown in [2] that in the orthogonal coordinates x, y the derivatives $\partial W/\partial x$, $\partial W/\partial y$ are related to the derivatives $\partial W/\partial x'$, $\partial W/\partial y'$ along arbitrarily directed axes x' and y', which make with the axis x the angles (x, x') and (x, y'), respectively, by the relations

$$\frac{\partial W}{\partial x} = \left(\frac{\partial W}{\partial x'}\frac{1}{\sin(x, x')} + \frac{\partial W}{\partial y'}\frac{1}{\sin(x, y')}\right) \Big/ [\text{ctan}(x, x') + \text{ctan}(x, y')], \tag{1}$$

$$\frac{\partial W}{\partial y} = -\left(\frac{\partial W}{\partial x'}\frac{1}{\cos(x, x')} - \frac{\partial W}{\partial y'}\frac{1}{\cos(x, y')}\right) \Big/ [\tan(x, x') + \tan(x, y')], \tag{2}$$

The expression for the projection of the gradient of the function W on the axis z in terms of the value of the derivatives along the axes x, y, and z' is determined by

$$\frac{\partial W}{\partial z} = -\left(\frac{\partial W}{\partial z'} - \cos(z', x)\frac{\partial W}{\partial x} - \cos(z', y)\frac{\partial W}{\partial y}\right) \Big/ \cos(z', z). \tag{3}$$

For a regularized grid Eq. (1) passes over to the equality $\partial W/\partial x = \partial W/\partial x'$ and (2) takes the form

$$\frac{\partial W}{\partial y} = -\left(\frac{\partial W}{\partial x}\text{ctan}(x, y') - \frac{\partial W}{\partial y'}\frac{1}{\sin(x, y')}\right). \tag{4}$$

The derivatives of the function w along the normal lines to the edges $x = x_{i+0.5,mj}$, $x = x_{i-0.5,mj}$, $y = y_{i,m+0.5,j}$, $y = y_{i,m-0.5,j}$, $z = z_{i,m,j+0.5}$, $z = z_{i,m,j-0.5}$ of a canonical element (a parallelepiped) constructed by the coordinate surfaces in the vicinity of the inner nodal point (x_{imj}, y_{mj}, z_j) are determined as follows. The derivative $\partial W/\partial x$ with respect to the coordinate x at the edge $x = x_{i+0.5,mj}$ is determined by a symmetric difference relation

$$W_{x,i+0.5,mj} = \frac{W_{i+1,mj} - W_{imj}}{h_{ximj}} \tag{5}$$

with an error of approximation of the order of h^2_{ximj}. The difference expression for the derivative $\partial W/\partial x$ at the nodal point (x_{imj}, y_{mj}, z_j) with an approximation error of the same order, which is based on (5), has the form

$$W_{x,imj} = \alpha_x W_{x,i+0.5,mj} + (1 - \alpha_x)\, W_{x,i-0.5,mj}, \tag{6}$$

where $\alpha_x = h_{x,i-1,mj}/(h_{x,imj} + h_{x,i-1,mj})$.

In [2] the derivative $\partial W/\partial y$ with respect to the coordinate y at the edge $y = y_{i,m+0.5,j}$ is determined in terms of the derivatives $\partial W/\partial x$ and $\partial W/\partial y'$ by a difference equation which approximates Eq. (4). In this case the axis y' with the origin at the point (x_{imj}, y_{mj}, z_j) passes through the nodal point $(x_{i'',m+1,j}, y_{m+1,j}, z_j)$ lying on the coordinate straight line $y = y_{m+1,j}$ at the smallest distance from the surface $x = x_{imj}$. The error of this method of determination of the derivative $\partial W/\partial y$ has an order $h_x^2 + h_y^2$ at the point of intersection of the edge $y = y_{i,m+0.5,j}$ of the canonical element by the axis y'. The error of approximation of the derivative $\partial^2 W/\partial y^2$ for the point (x_{imj}, y_{mj}, z_j) turns out to be proportional to the distance of the point of intersection from the coordinate plane $x = x_{imj}$. This error is substantial with a large degree of nonuniformity of the difference grid.

The drawback mentioned can be eliminated in the following way. On the coordinate straight line $y = y_{m+1,j}$ two neighboring nodal points $(x_{i'',m+1,j}, y_{m+1,j}, z_j)$ and $(x_{i''+1,m+1,j}, y_{m+1,j}, z_j)$ are selected, which lie closest to the coordinate surface $x = x_{imj}$. Identification of these points comes down to seeking an integral value of i'' proceeding from the requirement of satisfaction of the condition

$$|x_{i''m+1,j} - x_{imj}| + |x_{i''+1,m+1,j} - x_{imj}| = \min (|x_{s,m+1,j} - x_{imj}| + \\ + |x_{s+1,m+1,j} - x_{imj}|) \quad \text{for} \quad s = 1, 2, \ldots, I - 1). \tag{7}$$

Then difference approximations of Eq. (4) are constructed for two cases. In the first, the axis y with the origin at the point (x_{imj}, y_{mj}, z_j) passes through the nodal point $(x_{i''+1,m+1,j}, y_{m+1,j}, z_j)$, and in the second through the point $(x_{i''+1,m+1,j}, y_{m+1,j}, z_j)$. These approximations can be presented in the form

$$W'_{yi,m+0.5,j} = \frac{W_{i'',m+1,j} - W_{imj}}{h_{ymj}} + \frac{h'_{x,m+1,j}}{h_{ymj}} \frac{W_{imj} - W_{i-1,mj}}{h_{x,i-1,mj}}, \tag{8}$$

$$W''_{y,i,m+0.5,j} = \frac{W_{i''+1,m+1,j} - W_{imj}}{h_{ymj}} - \frac{h''_{x,m+1,j}}{h_{ymj}} \frac{W_{i+1,mj} - W_{imj}}{h_{ximj}}, \tag{9}$$

where $h'_{x,m+1,j} = x_{imj} - x_{i'',m+1,j}$; $h''_{x,m+1,j} = x_{i''+1,m+1,j} - x_{imj}$.

Approximating equations (8) and (9) are first multiplied by constant quantities subject to determination and then they are summed. The values of the constants are found from the condition that the error of approximation of the expression obtained as a result of summation for determination of the derivative $\partial W/\partial y$ at the point $(x_{imj}, y_{m+0.5}, z_j)$ of the edge $y = y_{m+0.5}$ of the considered canonical element be on the order of $h_{ximj}^2 + h_{ymj}^2$. As a result we find that

$$W_{y,i,m+0.5,j} = \frac{(W_{i'',m+1,j} - W_{imj})\, h''_{x,m+1,j} + (W_{i''+1,m+1,j} - W_{imj})\, h'_{x,m+1,j}}{h_{ymj}\,(h'_{x,m+1,j} + h''_{x,m+1,j})} - \frac{h'_{x,m+1,j}\, h''_{x,m+1,j}}{2h_{ymj}} W_{xx,imj}, \tag{10}$$

where

$$W_{xx,imj} = \frac{2}{h_{ximj} + h_{x,i+1,mj}} \left(\frac{W_{i+1,mj} - W_{imj}}{h_{x,i+1,mj}} - \frac{W_{imj} - W_{i-1,mj}}{h_{ximj}} \right). \tag{11}$$

By expanding the functions entering the formula into a Taylor series with respect to the central point of the edge $y = y_{i,m+0.5,j}$ of the considered canonical element, we see easily that the error of approximation for it is on the order of $h_x^2 + h_y^2$. It should be noted that when $h'_{x,m+1,j} = 0$ or $h''_{x,m+1,j} = 0$, i.e., when one of the points $(x_{i'',m+1,j}, y_{m+1,j}, z_j)$ or $(x_{i''+1,m+1,j}, y_{m+1,j}, z_j)$ lies in the plane $x = x_{imj}$, formula (10) becomes a symmetric difference expression similar to (5), which also has a second order of accuracy with respect to the mesh width of spatial division.

A six-point pattern for the arrangement of points in the plane $z = z_j$ corresponds to approximation (10): the derivative $\partial W/\partial y$ at the point $(x_{imj}, y_{m-0.5}, z_j)$ of the straight line $y = y_{m-0.5,j}$ is determined in terms of the values of the function W at five nodal points, three of which lie on the straight line $y = y_{mj}$ and two on the straight line $y = y_{m+1,j}$.

The difference expression of the derivative $\partial W/\partial y$ at the point $(x_{imj}, y_{m-0.5,j}, z_j)$ of the canonical element is given by a formula similar to (10):

$$W_{yi,m-0.5,j} = \frac{(W_{imj} - W_{i',m-1,j})\, h''_{x,m-1,j} + (W_{imj} - W_{i'+1,m-1,j})\, h'_{x,m-1,j}}{h_{y,m-1,j}\,(h'_{x,m-1,j} + h''_{x,m-1,j})} - \frac{h'_{x,m-1,j}\, h''_{x,m-1,j}}{2h_{y,m-1,j}} W_{xx,imj}. \quad (12)$$

Here $h'_{x,m-1,j} = x_{imj} - x_{i',m-1,j}$, $h''_{x,m-1,j} = x_{i'+1,m-1,j} - x_{imj}$.

The difference expressions for the derivatives $\partial W/\partial y$ and $\partial^2 W/\partial y^2$ at the nodal point (x_{imj}, y_{mj}, z_j) have the form

$$W_{yimj} = \alpha_y W_{yi,m+0.5,j} + (1 - \alpha_y)\, W_{yi,m+0.5,j}, \quad (13)$$

$$W_{yy,imj} = \frac{W_{y,i,m+0.5,j} - W_{y,i,m-0.5,j}}{y_{m+0.5,j} - y_{m-0.5,j}}, \quad (14)$$

where $\alpha_y = h_{y,m-1,j}/(h_{ymj} + h_{y,m-1,j})$.

The mixed derivative $\partial^2 W/\partial x \partial y$ at the point (x_{imj}, y_{mj}, z_j) is found by relations (5) and (6), where W_y is substituted for W.

The difference expression for the derivative $\partial W/\partial z$ at the point $(x_{imj}, y_{mj}, z_{j+0.5})$ of the edge $z = z_{j+0.5}$ of the canonical element, which has an error on the order of $h_x^2 + h_y^2 + h_z^2$, can be obtained in the following way. First, in the plane $z = z_{j+1}$ the two coordinate straight lines $y = y_{m'',j+1}$ and $y = y_{m''+1,j+1}$ at the smallest distances from the coordiate surface $y = y_{mj}$ are determined if

$$|y_{m''+1,j+1} - y_{mj}| + |y_{m'',j+1} - y_{mj}| = \min(|y_{g+1,j+1} - y_{mj}| + |y_{g,j+1} - y_{mj}|) \text{ for } g = 1, 2, \ldots, M-1. \quad (15)$$

On the straight line $y = y_{m'',j+1}$ the two neighboring nodal points $(x_{\bar{i},m'',j+1}, y_{m'',j+1}, z_{j+1})$, $(x_{\bar{i}+1,m'',j+1}, y_{m'',j+1}, z_{j+1})$ at the smallest distances from the coordinate surface $x = x_{imj}$ are found by a condition similar to (7). We denote as z' the axis lying on the line of intersection of the coordinate plane $x = x_{imj}$ with the plane containing the straight lines $y = y_{m'',j+1}$ and $y = y_{mj}$. The derivative of $W_{z'}$ along the axis z' at the point of its intersection with the plane $z = z_{j+0.5}$ is found, just as $w_{y,i,m+0.5,j}$, using a six-point pattern for $w_{\bar{i},m'',j+1}$, $w_{\bar{i}+1,m'',j+1}$ on the straight line $y = y_{m'',j+1}$, w_{imj}, $w_{i+1,mj}$, and $w_{i-1,mj}$ on the straight line $y = y_{mj}$:

$$W_{z'} = \frac{(W_{\bar{i},m'',j+1} - W_{imj})\, h''_{x,m'',j+1} + (W_{\bar{i}+1,m'',j+1} - W_{imj})\, h'_{x,m'',j+1}}{h^{(1)}\,(h'_{x,m'',j+1} + h''_{x,m'',j+1})} - \frac{h'_{x,m'',j+1}\, h''_{x,m'',j+1}}{2h^{(1)}} W_{xx,imj},$$

where $h'_{x,m'',j+1} = x_{imj} - x_{\bar{i},m'',j+1}$; $h''_{x,m'',j+1} = x_{\bar{i}+1,m'',j+1} - x_{imj}$; $h^{(1)} = [h^2_{z,j+1} + (y_{m'',j+1} - y_{mj})^2]^{0.5}$.

The derivative of $W_{z''}$ at the point of intersection of the planes $x = x_{imj}$, $z = z_{j+0.5}$ and the plane containing the straight lines $y = y_{m''+1,j+1}$ and $y = y_{my}$ is found in the same way. Finally, the derivative of $W_{zim,j+0.5}$ at the point $(x_{imj}, y_{mj}, z_{j+0.5})$ is found from the values of the derivatives $W_{z'}$, $W_{z''}$ and $W_{yy,imj}$

$$W_{zim,j+0.5} = \frac{W_{z'}\, h^{(1)}\, h''_{y,m'',j+1} + W_{z''}\, h^{(2)}\, h'_{y,m'',j+1}}{h_{z,j+1}\,(h'_{y,m'',j+1} + h''_{y,m'',j+1})} - \frac{h'_{y,m'',j+1}\, h''_{y,m'',j+1}}{2h_{z,j+1}} W_{yy,imj}. \quad (16)$$

Here $h'_{y,m'',j+1} = y_{mj} - y_{m'',j+1}$, $h''_{y,m'',j+1} = y_{m''+1,j+1} - y_{mj}$. The difference expression of $W_{z,i,m,j-0.5}$ of the derivative $\partial W/\partial z$ at the point $(x_{imj}, y_{imj}, z_{j-0.5})$ is constructed similarly. The derivatives $\partial W/\partial z$ and $\partial W^2/\partial z^2$ at the nodal point (x_{imj}, y_{mj}, z_j) are approximated by the expressions

$$W_{zimj} = \alpha_z W_{zim,j+0.5} + (1 - \alpha_z)\, W_{zim,j+0.5}\,, \tag{17}$$

$$W_{zz,imj} = (W_{zim,j+0.5} - W_{zim,j+0.5})/(z_{j+0.5} - z_{j-0.5})\,, \tag{18}$$

where $\alpha_z = h_{z,j-1}/(h_{zj} + h_{z,j-1})$. The mixed derivatives $\partial^2 W/\partial z\partial x$ and $\partial^2 W/\partial z\partial y$ at the nodal point (x_{imj}, y_{mj}, z_j) are determined by Eqs. (5), (6), and (10), (12), (13), respectively, into which the values of W_z calculated by (17) are substituted for W.

A mathematical model of processes of the transfer of energy, mass of substance, and momentum in a deformed body is based on the equations of thermoconcentrational elasticity [3, 4] in a quasistationary formulation, which for constant thermophysical and mechanical parameters can be written in the form

$$\frac{\partial \vartheta_g}{\partial t} = \sum_g a_{gs} \operatorname{grad}\ (\operatorname{div} \vartheta_s) + \Phi_g\,, \quad g, s = 1, 2, .., G\,, \tag{19}$$

$$\mu \nabla^2 U + (\lambda + \mu) \operatorname{grad} (\operatorname{div} U) - (\lambda + 2\mu/3) \operatorname{grad} (N) + F = 0\,. \tag{20}$$

Here N is the function of variation of a specific volume of the body in its free expansion caused by variation of temperature and concentration of components [4], $N = \sum_s \alpha_s(\vartheta_s - \vartheta_{s0})$, $\alpha_s = dq/d\vartheta_s/q_0$.

Boundary conditions of heat and mass transfer of the first, second, or third kind and initial conditions are assigned for Eq. (19). For Eq. (20) the conditions on the circuit are assigned by the displacement function U or in the form of a vector of external stress p, the projections of which p_x, p_y, p_z on the axis x, y, z are related to internal stresses by relationships of the form [4, 5]

$$p_x = \sigma_{xx} \cos (x, n) + \sigma_{xy} \cos (y, n) + \sigma_{xz} \cos (z, n)\,,$$

where $\sigma_{xx} = (2\mu + \lambda)\partial u/\partial x + \lambda(\partial v/\partial y + \partial w/\partial z) - (\lambda + 2\mu/3)N$; $\sigma_{xy} = \mu(\partial v/\partial x + \partial u/\partial y)$.

A difference approximation of Eq. (19) is constructed using a three-layer explicit difference scheme [4, 6]:

$$\frac{\vartheta_{gimj}^{n+1} - \vartheta_{gimj}^{n}}{h_t}(1 + b_{gimj}) - \frac{\vartheta_{gimj}^{n} - \vartheta_{gimj}^{n-1}}{h_t} b_{gimj} = \sum_g a_{gs} (\vartheta_{sxx,imj} + \vartheta_{syy,imj} +$$

$$+ \vartheta_{szz,imj}) + \Phi_g\,, \quad g, s = 1, 2, ..., G\,; \quad b_{gimj} \geq 0\,. \tag{21}$$

After an arbitrary choice of the mesh widths of the difference grid h_{ximj}, h_{ymj}, h_{zj}, and h_t the values of the parameter b_{gimj} are determined in accordance with the condition of stability of Eq. (21)

$$b_{gimj} = 0.5\,(h_t/\Delta_{timj} - 1) \quad \text{when} \quad h_t > \Delta_{timj}\,; \quad b_{gimj} = 0 \quad \text{when} \quad h_t \leq \Delta_{timj}\,. \tag{22}$$

Here $\Delta_{timj} = 1/[2a_{gg}(1/h_{ximj}^2 + 1/h_{ymj}^2 + 1/h_{zj}^2)]$.

Approximations of the equation of conservation of momentum (20), solved by a time-dependent technique, are constructed similarly in projections onto the axes x, y, and z. For projection (20) onto the axis x the approximation has the form

$$\frac{u_{imj}^{n+1} - u_{imj}^{n}}{h_{tu}}(1 + b_{uimj}) - \frac{u_{imj}^{n} - u_{imj}^{n-1}}{h_{tu}} b_{uimj} = (2\mu + \lambda)\, u_{xx,imj} + \mu u_{yy,imj} +$$

$$+ (\mu + \lambda)\, v_{xy,imj} + (\mu + \lambda)\, W_{xz,imj} - (\lambda + 2\mu/3)\, N_{x,imj} + X\,, \quad b_{uimj}^{n} \geq 0\,. \tag{23}$$

The necessary condition for stability of Eq. (23) is similar to that for (22), with $\Delta_{tim} = 1/\{2[(2\mu + \lambda)/h_{ximj}^2 + \mu/h_{umj}^2 + \mu/h_{zj}^2]\}$.

On the basis of the method of canonical elements, with allowance for the above difference approximations of partial derivatives and differential equations, we developed an algorithm for modeling processes of transfer in multiply connected systems of solid bodies of an arbitrary shape. The position of the boundaries of the region can be assigned analytically or by a table of coordinates of a certain number of boundary points, on the basis of which the coordinates of boundary nodal points and direction cosines of outer normals at these points are determined by a special subprogram. Before calculating the grid functions, the arrays of the coordinates of nodal points of the region, the weight parameters b_{imj}, and the values of i' and i'' for two-dimensional problems and additionally of the values of m', m'', i'_{j-1}, i''_{j-1}, i'_{j+1} for three-dimensional problems are constructed by special subprograms.

To judge the efficiency and accuracy of the formulated technique we solved numerically, using the program complex, some problems of heat conduction and deformation in regions with curvilinear boundaries for which an accurate solution can be obtained. These problems include, in particular, an axisymmetric problem of heat and mass transfer for an unbounded hollow cylinder made of a material with constant thermophysical characteristics, which in cylindrical coordinates is reduced to a one-dimensional problem; this problem has an accurate analytical solution under the first-, second,- and third-kind boundary conditions of heat transfer [3, 7]. In the solution of two-dimensional doubly-connected problems of transfer for a hollow cylinder $r_0 \le r \le R$ in Cartesian coordinates, the mesh width along the axis y was considered uniform in the regions $0 \le y \le R - r_0$, $R - r_0 \le y \le R + r_0$, and $R + r_0 \le y \le 2R$. Two nodal points lay at a small distance on the coordinate lines $y = 0$ and $y = 2R$. I nodal points lay on the remaining coordinate lines $y = y_m$ ($m = 2, 3, ..., M - 1$) of the difference grid. The mesh width h_{xm} was considered uniform on the segmets $0 < y < R - r_0$ and $R + r_0 < y < 2R$ and also between the surfaces $r = r_0$ and $r = R$ on the segment $R - r_0 \le y \le R + r_0$. On the straight lines $y = R - r_0$ and $y = R + r_0$ the position of the surface $r = r_0$ was determined by two closely lying nodal points. The mean deviation of the values of relative temperature found numerically from an accurate analytical solution at $r_0 = R/2$, $I = 10$, $M = 17$ and boundary condition of the third kind was $\overline{\Pi} = 0.41\%$, and the maximum was $\Pi_{\max} = 2.9\%$; $\overline{\Pi} = 0.23\%$ and $\Pi_{\max} = 2.2\%$ at $I = 20$ and $M = 33$.

In adjustment of the problem and estimation of the accuracy of the results of calculation of momentum transfer, as a standard we used an accurate analytical solution of an axisymmetric stationary problem of the stressed state of a hollow cylinder (this state being caused by nonuniformity of the fields of temperature and concentration of components and also by the effect of uniformly distributed pressures p_0 on the inner cylindrical surface of radius $r = r_0$ and p on the outer surface of radius $r = R$ and of a resultant force P_z along the cylinder axis z), which can be presented in the form

$$U(r) = \frac{1}{r(1-\nu)} \left[(1+\nu) \int_{r_0}^{r} Nrdr + \frac{r^2(1-3\nu) + r_0^2(1+\nu)}{R^2 - r_0^2} \int_{r_0}^{R} Nrdr + \right.$$

$$\left. + \frac{r\nu P_z}{E\pi(R^2 - r_0^2)} \right] + \frac{1}{Er(R^2 - r_0^2)} [r^2(1-\nu)(p_0 r_0^2 - pR^2) + (1+\nu) r_0^2 R^2 (p_0 - p)], \tag{24}$$

$$\varepsilon_z = \frac{2}{R^2 - r_0^2} \int_{r_0}^{R} Nrdr - \frac{2\nu}{E} \frac{p_0 r_0^2 - pR^2}{R^2 - r_0^2} - \frac{P_z}{E\pi(R^2 - r_0^2)}. \tag{25}$$

Here U is the function of displacement of the points of the cylinder in a radial direction; ε_z is the relative elongation along the axis z. It should be noted that at $r_0 = 0$ relations (24), (25) are a solution of the problem of thermoconcentrational elasticity for a solid cylinder of radius R. On the basis of the solution (24), (25) and the corresponding conversion formulas we found the components of the vector of transfer and tensors of deformations and stresses for solid and hollow cylinders in Cartesian coordinates.

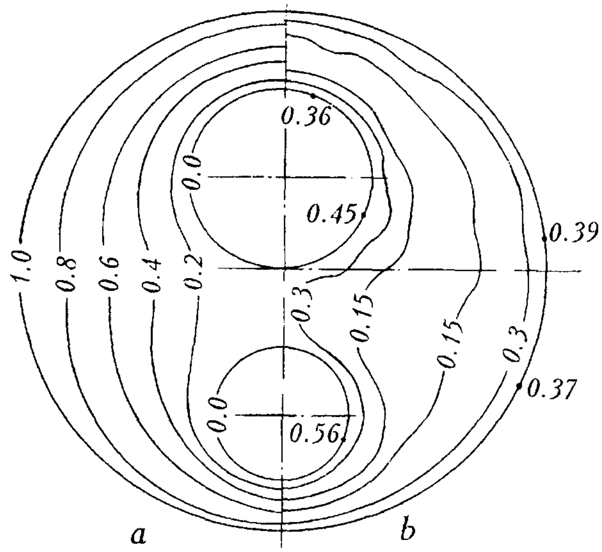

Fig. 1. Distribution of relative values of temperature (a) and generalized stress (b) in an iron cylinder with two channels.

The mean error of the numerical solution of problems of momentum transfer in a hollow cylinder, when the function U is given for the boundary points, was $\overline{\Pi} = 0.82\%$ at $I = 10$ and $M = 17$, and with assignment of external stresses p, p_0, and p_z the error was $\overline{\Pi} = 4.2\%$.

Figure 1 presents, as an example, the results of calculation using the program complex of the fields of relative temperatures $\overline{T} = (T - T_{in})/(T_{out} - T_{in})$ and generalized stresses $\bar{\sigma} = \sigma/|E(T_{out} - T_{in})|\cdot 10^5$ in an iron cylinder of diameter d with two channels of diameters $0.35d$ and $0.25d$. It should be noted that a change in the configuration of the region leads only to reassignment of the arrays of the coordinates of its boundary points.

Numerical experiments indicate the efficiency and high accuracy of the presented technique and the possibility of constructing on its basis a unique program complex for simulating transfer processes in multiply connected systems of an arbitrary configuration with variable thermophysical characteristics for arbitrary initial and boundary conditions.

NOTATION

ϑ, temperature or concentration of the component; U, vector of transfer with projections u, v, and w on the axes of coordinates x, y, and z; μ, λ, Lamé coefficients; E, elasticity modulus; ν, Poisson coefficient; N, function of variation of the specific volume of the body; F, mass force with projections X, Y, and Z on the axes x, y, z; p, vector of external stress with projections p_x, p_y, p_z on the axes x, y,z; R, values of the radius r for the inner and outer surfaces of the cylinder; t, time; h_{ximj}, h_{yim}, h_{zj}, h_{in}, mesh widths of the difference grid along the coordinate axes x, y, z, tt; Φ, density of the sources of heat and mass. Subscripts: in, inner; out, outer.

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